

Non-conservation of Carter in black hole spacetimes

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Abstract.

Freely falling point particles in the vicinity of Kerr black holes are subject to a conservation law, that of their Carter constant. We consider the conjecture that this conservation law is a special case of a more general conservation law, valid for arbitrary processes obeying local energy momentum conservation. Under some fairly general assumptions we prove that the conjecture is false: there is no conservation law for conserved stress-energy tensors on the Kerr background that reduces to conservation of Carter for a single point particle.

The validity of conservation laws in physics can depend on the type of interactions being considered. For example, baryon number minus lepton number, $B - L$, is an exact conserved quantity in the Standard Model of particle physics. However, it is no longer conserved when one enlarges the set of interactions under consideration to include those of many grand unified theories [1], or those that occur in quantum gravity [2].

The subject of this note is the conserved Carter constant for freely falling point particles in black hole spacetimes [3]. As is well known, this quantity is not associated with a spacetime symmetry or with Noether's theorem. Instead it is obtained from a symmetric tensor K_{ab} which obeys the Killing tensor equation $\nabla_{(a}K_{bc)} = 0$ [4], via $K = K_{ab}k^ak^b$, where k^a is four-momentum. Is this conservation law specific to freely falling particles, or does it persist in the presence of more general interactions?

The Carter constant conservation law has already been generalized in a variety of directions:

- It applies to charged particles in rotating charged black hole spacetimes [3].
- It has been generalized to spinning test particles, to linear order in the spin [5] ‡.
- For a free scalar field on the Kerr background, Carter showed that the differential operator $\mathcal{D} = K^{ab}\nabla_a\nabla_b + (\nabla_a K^{ab})\nabla_b$ commutes with the d'Alembertian, which implies the existence of a conserved quantity for the field [8]. For a free complex scalar field Φ , the charge associated with the conserved current $j^a = i[(\mathcal{D}\Phi)^*\nabla^a\Phi - (\nabla^a\mathcal{D}\Phi)^*\Phi]/2$ is a generalization of the Carter constant for particle motion, in the following sense. Solutions of the form $\Phi \propto \exp[i\varphi/\varepsilon]$ in the eikonal limit $\varepsilon \rightarrow 0$ can be interpreted as streams of particles, and the conserved charge is just the sum of the Carter constants of the particles. This is valid for both massive and massless fields.
- There is also a conserved quantity for spin 1/2 fields related to the Killing tensor. On the Kerr spacetime the Killing tensor can be expressed as the square of an antisymmetric Killing-Yano tensor f_{ab} , for which $\nabla_{(a}f_{b)c} = 0$ and $K_{ab} = f_{ac}f_b{}^c$ [9, 10]. Carter and McLenaghan [11] showed that the operator $i\gamma_5\gamma^a(f_a{}^b\nabla_b - \gamma^b\gamma^c\nabla_cf_{ab}/6)$ commutes with the Dirac operator and so gives rise to a conserved quantity. A similar construction for spin-1 fields can be found in Ref. [12].
- Recently, Ashtekar and Kesavan have shown that in spacetimes which settle down at late times to a Kerr black hole, the Killing tensor at future null infinity can be expressed as a linear combination of products of asymptotic symmetry vector fields (BMS generators), allowing them to compute a charge associated with any cut and derive an asymptotic conservation law [13].
- There is no known general local conservation law associated with the Killing tensor for spin 2 fields in Kerr, that is, for linearized vacuum perturbations. However, there are hints that a conserved current may exist in this case. Specifically, for the radiation-reaction inspirals of point particles into black holes, it is possible to compute the time averaged time derivative of the particle's Carter constant [14, 15, 16, 17]. The result consists of two terms, a term involving the

‡ The motion of a spinning point particle in the Kerr spacetime is thus integrable to linear order in spin [6]. This does not contradict the fact that chaotic behavior is seen in numerical studies of spinning point particle dynamics [7], since that behavior is due to effects that are higher order in spin.

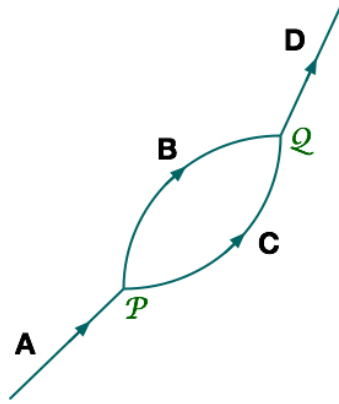


Figure 1. An illustration of a process in which local energy momentum conservation is satisfied but in which Carter constant conservation is violated. A particle A is freely falling in the Kerr spacetime. At some point \mathcal{P} it splits into two particles B and C . Those particles then freely fall, and collide at some later event \mathcal{Q} to form a single particle D . The Carter constants of A and D do not coincide.

amplitudes of the gravitational wave modes at future null infinity that is naturally interpreted as a flux to infinity, and a term involving the amplitudes of the modes at the horizon that is naturally interpreted as a flux down the horizon.

In this note we consider a different possible type of generalization of the conservation law, the possibility that the Carter quantity may be conserved under local interactions between particles that obey local stress energy conservation. Specifically, suppose we are given a conserved symmetric tensor T_{ab} on the Kerr spacetime, with compact spatial support. Does there exist a quantity \mathcal{K}_Σ which can be computed from T_{ab} and its derivatives on any Cauchy surface Σ , which has the properties that (i) \mathcal{K}_Σ is independent of Σ , and (ii) \mathcal{K}_Σ reduces to the Carter constant for a single point particle? For example, one could consider quadratic functionals of the form[§]

$$\mathcal{K}_\Sigma = \int_\Sigma d^3\Sigma_a(x) \int_\Sigma d^3\Sigma_{a'}(x') T^{ab}(x) K_{bb'}(x, x') T^{a'b'}(x'). \quad (1)$$

Such functionals satisfy property (ii) if the bitensor $g_a{}^{b'}(x, x') K_{bb'}(x, x')$ reduces to the Killing tensor K_{ab} in the coincidence limit $x' \rightarrow x$ (see Appendix B). Of course, they do not necessarily satisfy property (i). If a conserved quantity of this type existed, one could say that the Carter constant is conserved not just for freely falling, non-interacting particles, but also under general processes that obey local stress energy conservation.

No such conservation law exists. There is a very simple argument which shows this, which is as follows. Consider a freely falling particle A in the Kerr spacetime. Suppose that at some point \mathcal{P} , it splits into two particles B and C , obeying energy-momentum conservation. Thus the four-momenta \vec{k} of the particles are related by

$$\vec{k}_A(\mathcal{P}) = \vec{k}_B(\mathcal{P}) + \vec{k}_C(\mathcal{P}). \quad (2)$$

[§] The functional (1) is intended as an illustrative example only; the argument below is not restricted to functionals of this form.

Suppose that the initial four-momenta and masses of B and C are carefully chosen so that these two particles, after falling freely for some time, come together again at some other spacetime point \mathcal{Q} . At that point the two particles combine to form a fourth particle, D , again obeying energy-momentum conservation:

$$\vec{k}_D(\mathcal{Q}) = \vec{k}_B(\mathcal{Q}) + \vec{k}_C(\mathcal{Q}). \quad (3)$$

Between the collisions, the Carter constants of the individual particles $K_A = K_{ab}k_A^a k_A^b$, etc, of the individual particles are locally conserved.

Now, if a general conservation law of the type discussed above existed, then at early times the putative conserved quantity \mathcal{K} would be simply the Carter constant of A , i.e

$$\begin{aligned} \mathcal{K} &= K_A = K_{ab}(\mathcal{P})k_A^a(\mathcal{P})k_A^b(\mathcal{P}) \\ &= K_B + K_C + 2K_{ab}(\mathcal{P})k_B^a(\mathcal{P})k_C^b(\mathcal{P}). \end{aligned} \quad (4)$$

Similarly at late times^{||} it would be

$$\begin{aligned} \mathcal{K} &= K_D = K_{ab}(\mathcal{Q})k_D^a(\mathcal{Q})k_D^b(\mathcal{Q}) \\ &= K_B + K_C + 2K_{ab}(\mathcal{Q})k_B^a(\mathcal{Q})k_C^b(\mathcal{Q}). \end{aligned} \quad (5)$$

However, by numerically integrating the geodesic equations in Kerr, it is easy to find examples of scenarios of this kind where the initial and final values (4) and (5) do *not* coincide (see Appendix A). This occurs whenever the cross term $K_{ab}k_B^a k_C^b$ has different values at \mathcal{P} and \mathcal{Q} .

In conclusion, generalizations of the Carter constant conservation law are relevant to the program of computing gravitational wave signals from point particles inspiralling into spinning black holes[18, 19, 20, 21]. One might have hoped for a unified conservation law for “Carter”, analogous to that for energy, that some combination of the Carter constant of the particle and the “Carter” in the gravitational wave field be conserved. The result presented here suggests that no such unified law exists.

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Appendix A. Numerical Example

We use Boyer-Lindquist coordinates (t, r, θ, ϕ) , choose units so that the mass of the black hole is $M = 1$, and choose the value of the spin parameter to be $a = 0.84$. The event \mathcal{P} is $(0, 3.508712731567, \pi/2, 0)$, and \mathcal{Q} is $(48.77745940108, 3.533582213837, 1.549703133614, 7.832144679602)$. The contravariant components of B ’s four-velocity at \mathcal{P} are

$$\vec{u}_B(\mathcal{P}) = (2.043684586293, 0.00004124671403569, 0.2549376720024, 0.1436563472919),$$

^{||} At intermediate times, between the two collisions, the value of the conserved quantity \mathcal{K} would depend on the details of how the conserved quantity is supposed to be computed. For example, in the model (1) it would depend on values of the bitensor $K_{ab'}$ away from the coincidence limit. This is why we need to consider a process that begins and ends with a single particle, in order to get a clean test of the conjecture.

while those of C are

$$\vec{u}_C(\mathcal{P}) = (2.083715919305, 0.01417859756406, -0.2712642321094, 0.1252970976976).$$

These four-velocities were chosen to be close to the case where $u_C^\theta = -u_B^\theta$ and re-intersection is guaranteed (although one can show that $K_A = K_D$ in this special case). We choose the masses of B and C to be unity, so that $\vec{k}_B \cdot \vec{k}_B = \vec{k}_C \cdot \vec{k}_C = -1$. The interval of proper time from \mathcal{P} to \mathcal{Q} for B is $\Delta\tau_B = 20.424435554150$, while for C it is $\Delta\tau_C = 24.10742349604$. Our numerical integration code conserves the quantities E , L_z , and K along each geodesic to within one part in 10^{12} , and each geodesic reaches the event \mathcal{Q} to within one part in 10^{12} as well. With these choices the initial Carter constant is $K_A = 0.04154646396564$, while the final one is $K_D = 0.04392127426890$. The fractional difference is

$$\frac{2(K_A - K_D)}{K_A + K_D} = -0.055572087 \pm 0.000000002,$$

which is nonzero.

We have also constructed several other numerical examples. Those examples are sufficient to prove a slight generalization of the result in the body of the paper: there is no general conservation law for conserved stress-energy tensors associated with any Killing tensor of the form $K_{ab} + \lambda g_{ab}$ for any value of λ .

Appendix B. Observables that reduce to Carter constant for point particles

In this appendix we show that the expression (1) reduces to the Carter constant of a point particle when the stress energy tensor is taken to be that of a point particle. If the particle's worldline is written as $x^\alpha = z^\alpha(\lambda)$, then this stress energy tensor is

$$T^{\alpha\beta}(x) = \int d\lambda p^\alpha(\lambda) p^\beta(\lambda) \frac{\delta^{(4)}(x - z(\lambda))}{\sqrt{-g}}, \quad (\text{B.1})$$

where $p^\alpha = dz^\alpha/d\lambda$. We now insert this into the expression (1). We use Gaussian normal coordinates adapted to the surface Σ , so that the surface is $t = t_0$, the metric is $ds^2 = -dt^2 + h_{ij}dx^i dx^j$, and $d^3\Sigma_a = \sqrt{h}n_a d^3x$ with $\vec{n} = \partial_t$ the unit normal. The result is

$$\begin{aligned} \mathcal{K}_\Sigma = & \int dt d^3x \sqrt{h} n_\alpha \int dt' d^3x' \sqrt{h'} n_{\alpha'} \int d\lambda \int d\lambda' K_{\beta\beta'}[z(\lambda), z(\lambda')] \\ & \times p^\alpha(\lambda) p^\beta(\lambda) p^{\alpha'}(\lambda') p^{\beta'}(\lambda') \delta[t - t(\lambda)] \delta[t' - t(\lambda')] \frac{\delta^3[\mathbf{x} - \mathbf{z}(\lambda)]}{\sqrt{h}} \frac{\delta^3[\mathbf{x}' - \mathbf{z}(\lambda')]}{\sqrt{h'}}. \end{aligned} \quad (\text{B.2})$$

Evaluating the time integrals of the delta functions give factors of $|dt/d\lambda| = (n_\alpha p^\alpha)^{-1}$, and so \mathcal{K}_Σ reduces to $p^\beta p^\gamma K_{\beta\gamma}$ evaluated at the point where the worldline crosses the surface. Since we are assuming that the bitensor reduces to the Killing tensor in the coincidence limit, this is just the Carter constant.

References

- [1] H. Georgi and S.L. Glashow. Unity of All Elementary Particle Forces. *Phys.Rev.Lett.*, 32:438–441, 1974.

- [2] S.W. Hawking, D.N. Page, and C.N. Pope. The propagation of particles in spacetime foam. *Physics Letters B*, 86(2):175 – 178, 1979.
- [3] Brandon Carter. Global structure of the kerr family of gravitational fields. *Phys. Rev.*, 174:1559–1571, Oct 1968.
- [4] M. Walker and R. Penrose. On quadratic first integrals of the geodesic equations for type { 22 } spacetimes. *Communications in Mathematical Physics*, 18:265–274, December 1970.
- [5] R. Rudiger. Conserved Quantities of Spinning Test Particles in General Relativity. II. *Royal Society of London Proceedings Series A*, 385:229–239, January 1983.
- [6] Tanja Hinderer, Alessandra Buonanno, Abdul H. Mrou, Daniel A. Hemberger, Geoffrey Lovelace, et al. Periastron advance in spinning black hole binaries: comparing effective-one-body and Numerical Relativity. *Phys.Rev.*, D88(8):084005, 2013, 1309.0544.
- [7] Kenta Kiuchi and Kei-ichi Maeda. Gravitational waves from chaotic dynamical system. *Phys.Rev.*, D70:064036, 2004, gr-qc/0404124.
- [8] Brandon Carter. Killing tensor quantum numbers and conserved currents in curved space. *Phys. Rev. D*, 16:3395–3414, Dec 1977.
- [9] Roger Penrose. Naked singularities. *Annals of the New York Academy of Sciences*, 224(1):125–134, 1973.
- [10] Roger Floyd. *The dynamics of Kerr fields*. PhD thesis, University of London, 1974.
- [11] B. Carter and R. G. McLenaghan. Generalized total angular momentum operator for the dirac equation in curved space-time. *Phys. Rev. D*, 19:1093–1097, Feb 1979.
- [12] G. F. Torres del Castillo. Killing spinors and massless spinor fields. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 400(1818):119–126, 1985.
- [13] A. Ashtekar and A. Kesevan. unpublished. *Presentation at Cracow School of Theoretical Physics, LIII Course, 2013, available at <http://th-www.if.uj.edu.pl/school/2013/talks/Kesavan.pdf>*.
- [14] Yasushi Mino. Perturbative approach to an orbital evolution around a supermassive black hole. *Phys. Rev.*, D67:084027, 2003, gr-qc/0302075.
- [15] S. Drasco, É. É. Flanagan, and S. A. Hughes. Computing inspirals in Kerr in the adiabatic regime: I. The scalar case. *Classical and Quantum Gravity*, 22:801, August 2005, gr-qc/0505075.
- [16] N. Sago, T. Tanaka, W. Hikida, and H. Nakano. Adiabatic Radiation Reaction to Orbits in Kerr Spacetime. *Progress of Theoretical Physics*, 114:509–514, August 2005, arXiv:gr-qc/0506092.
- [17] N. Sago, T. Tanaka, W. Hikida, K. Ganz, and H. Nakano. Adiabatic Evolution of Orbital Parameters in Kerr Spacetime. *Progress of Theoretical Physics*, 115:873–907, May 2006, arXiv:gr-qc/0511151.
- [18] S. Detweiler. Perspective on gravitational self-force analyses. *Classical and Quantum Gravity*, 22:681, August 2005, gr-qc/0501004.
- [19] E Poisson, A Pound, and I Vega. The Motion of Point Particles in Curved Spacetime. *Living Reviews in Relativity*, 14:7, September 2011, 1102.0529.
- [20] L Barack. TOPICAL REVIEW: Gravitational self-force in extreme mass-ratio inspirals. *Classical and Quantum Gravity*, 26(21):213001, November 2009, 0908.1664.
- [21] Leor Barack. Gravitational self-force: Orbital mechanics beyond geodesic motion. In Ji Bik and Tom Ledvinka, editors, *General Relativity, Cosmology and Astrophysics*, volume 177 of *Fundamental Theories of Physics*, pages 147–168. Springer International Publishing, 2014.